Toric Structure in Statistical Models through Symmetry Lie Algebra

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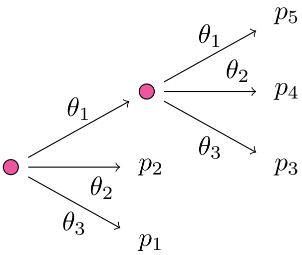

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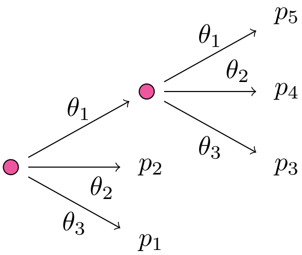

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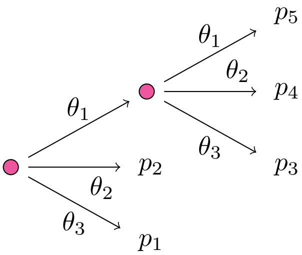

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$$p_5 \mapsto \theta_1^2$$

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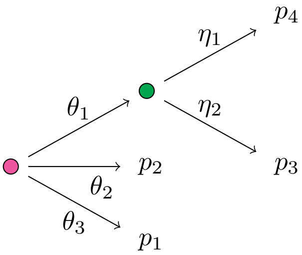

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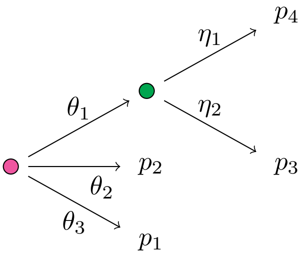

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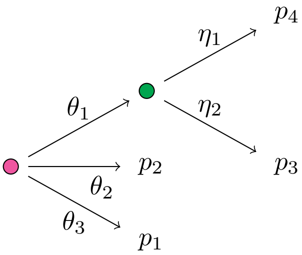

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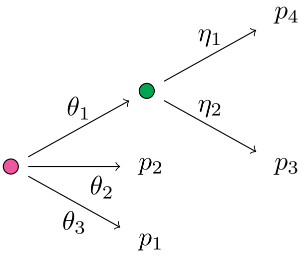

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- φ is surjective.
- Variety cut out by Ker(φ) Toric?

Why Toric?

1. Binomial generators give Markov basis, contribute in hypothesis testing.

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3. The rich algebra, geometry, and combinatorics of a toric ideal facilitates computations on the maximum likelihood degree and estimate
   1. Steven Evans and Terrence Speed. *"Invariants of some probability models used in phylogenetic inference."* The Annals of Statistics (1993).
   2. Carlos Amendola, Dimitra Kosta, Kaie Kubjas. *"Maximum Likelihood Estimation of Toric Fano Varieties"* (2020): no.11.1, 15-30.

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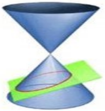

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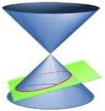

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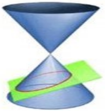

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Let $g \in \mathfrak{gl}_{I_d}$, $p \in I = (p_1, \cdots, p_k)$, where $p_1, \cdots, p_k$ generates I, and $\deg(p_i) = d$.

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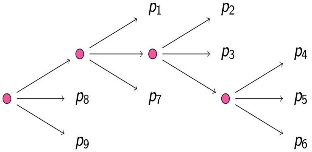

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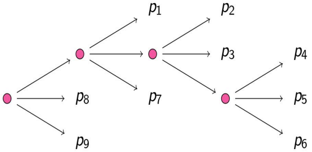

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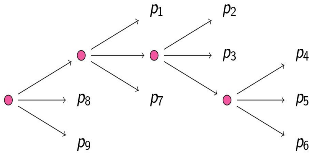

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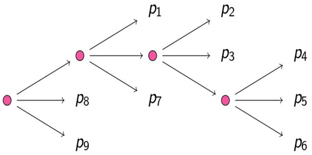

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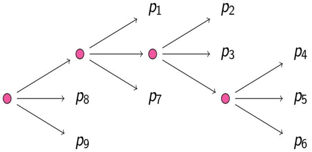

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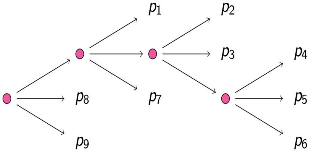

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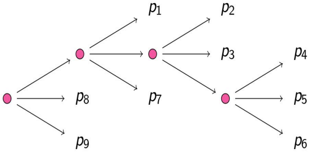

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- $\implies$ Conjecture is FALSE!

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Found: $\dim(V_I) = 4$, $\dim(\mathfrak{gl}_I) = 5$.

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$$-\tfrac{\tilde{q_1}+\tilde{q_2}}{2} = ix_2^2 + x_1x_6, \qquad \tfrac{\tilde{q_2}-\tilde{q_1}}{2} = x_4x_6 - ix_3^2$$

Thank you

Thank you!

References

Christiane Görgen, Aida Maraj, and Lisa Nicklasson
Staged tree models with toric structure,
Journal of Symbolic Computation, Volume 113, 2022, Pages 242-268, ISSN 0747-7171, https://doi.org/10.1016/j.jsc.2022.04.006.