Tensors of Minimal Border Rank

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Simultaneously Diagonalizable Matrices

Let $M_1, \cdots, M_n$ be n simultaneously diagonalizable $m \times m$ matrices.

Classical Problem

Classify the closure of the space of a finite number of simultaneously diagonalizable matrices.

Known Results

- End Closed Condition: Known since 1960.[Gerstenhaber]
- Flag Condition: Appeared in 1997.[Burgisser, Clausen, Shokrollahi]

Definitions

Let $T \in A \otimes B \otimes C$.
T can be seen as a linear map $T_A : A^* \to B \otimes C$. Similarly T_B and T_C.

Definition

The *rank* of a tensor T, denoted $\mathbf{R}(T)$, is the smallest r such that $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$.

Definition

The *border rank* of a tensor T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that $T = \lim_{\epsilon \to 0} T_\epsilon$ where $\mathbf{R}(T_\epsilon) = r$.

Definition

The tensor T is called *concise* if T_A, T_B and T_C are of full rank.

Questions

Assume $\dim(A) = \dim(B) = \dim(C) = m$ and $T \in A \otimes B \otimes C$.

Question 1

Find equations for the set of border rank at most m tensors.

Question 1 is answered up to dimension m=4.[Friedland]

Assuming T is concise, minimal possible border rank for T is m.

Question 2

Find equations for the set of concise minimal border rank tensors.

Under a natural genericity condition this question is same as characterizing the closure of simultaneously diagonalizable matrices.

Questions

$B \otimes C$ is identified with linear maps from B^* to C, Hom(B^*, C).

Definition

T is called 1_A-*generic*(similarly 1_B and 1_C) if there exist $x \in A^*$ such that $T_A(x)$ is invertible. T is called 1_*-*generic* if it is 1_A, 1_B or 1_C-generic.

If T is 1_*-generic then we reduce to the previous classical problem of classifying the closure of simultaneously diagonalizable matrices.

Question 3

Find equations for the set of concise, 1_*-generic, minimal border rank tensors.

Question 3 is answered for $m = 5$.[Landsberg, Michalek]

Motivation

- Question 1 is finding equations for the secant variety, $\sigma_m(\mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1})$.
- Complexity Theory: Latest bound on the exponent of matrix multiplication is achieved through Coppersmith-Winograd tensor. Which is a concise minimal border rank tensor.
- Classical Linear Algebra: Closure of simultaneously diagonalizable matrices.
- Algebraic Geometry: Hilbert schemes and Quot Schemes.

Strassen's Equations

Simultaneously diagonalizable $\implies$ Commuting.

Theorem ([Strassen])

Let X_1, X_2 and Y be in $T_A(A^*)$. If T is of minimal border rank then $\mathit{adj}(Y)X_1\mathit{adj}(Y)X_2 - \mathit{adj}(Y)X_2\mathit{adj}(Y)X_1 = 0$.

Assuming T 1_*-generic, we can take Y to be of full rank and in particular identity matrix. Then Strassen's Equations precisely reduces to commuting criterion of X_1 and X_2.

These are necessary conditions a tensor must satisfy in order to be of minimal border rank.

Koszul Flattenings

Let $T \in A \otimes B \otimes C$ and $\dim(A) = \dim(B) = \dim(C) = m$.

$$T^{\wedge p} : B^* \otimes \bigwedge^p A \xrightarrow{T_B \otimes Id} A \otimes \bigwedge^p A \otimes C \xrightarrow{\pi \otimes Id} \bigwedge^{p+1} A \otimes C$$

Theorem ([Landsberg, Ottaviani], Thm 2.1)

$\underline{R}(T) \geq \frac{rank(T^{\wedge p})}{\binom{m-1}{p}}$

Remark: $p = 1$ Koszul flattening equations are stronger than Strassen's equations.

Towards m=5 and 6, 1_A-generic case

In this case T is 1_A-generic and concise.

- Already solved, Question 3 for m=5.[Landsberg, Michalek]
- Known answer: Strassen's equations together with end closed condition.

Theorem (J. Jelisiejew, K. Sivic)

The closure of the space of 4-tuple of 5×5 commuting matrices is not irreducible and has exactly two components.
The principal component is the closure of simultaneously diagonalizable matrices and one other bad component.

Upshot: This extends to $m = 6$ and the same remains true!

Progress

Theorem (Jelisiejew,Landsberg,Pal)

Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, where $m = 5, 6$, be a concise 1_*-generic tensor. Then T is of minimal border rank if and only if T satisfies Strassen's equations and End Closed condition.

Towards m=5, 1-degenerate

Lemma ([M. D. Atkinson])

Let $E \in \mathbb{C}^m \otimes \mathbb{C}^m$ *be bounded rank* r*. Then we may choose bases such that there is* $X_1 \in E$ *of the form*

$$X_1 = \begin{pmatrix} \mathit{Id}_r & 0 \\ 0 & 0 \end{pmatrix}$$

and for any $X \in E$*, with the same blocking,*

$$X = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

and satisfies $CA^kB = 0$ *for all* $k \geq 0$*.*

Remark: Above lemma + Strassen's equations $\implies$ Friedland's Normal Form.

Work in progress

- Helped us write down a family of 1-degenerate (concise) minimal border rank tensors, thanks to Prof Michalek for border rank decomposition.
- FNF+$p = 1$ Koszul Flattening+ Flag condition leaves finitely cases to check/ prove.

Expectation: $p = 1$ Koszul flattening equations will be sufficient.

This is joint work(in progress) with Prof. JM Landsberg and Prof. Joachim Jelisiejew.

Thank you

Thank you!

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Explanation

Let T be concise, 1_A-generic, and $\alpha_1, \cdots, \alpha_m$ be a basis of A^* with $T_A(\alpha_1)$ full rank. Consider $Id_{B^*}, T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m) \in \mathrm{Hom}(B^*, B^*)$.

If T is of rank m then note that $T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m)$ needs to be simultaneously diagonalizable matrices. This is basically consequence of Strassen's equations.

Thus if T is of border rank m then $T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m)$ has to be in the closure of the space of simultaneously diagonalizable tuple of matrices.

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End Closed Condition

End Closed Condition

Let $T \in A \otimes B \otimes C$ be a concise 1_A-generic tensor and $\alpha \in A^*$ such that $T_A(\alpha)$ has full rank. Then $T(\alpha)^{-1}T_A(A^*)$ is a subalgebra of $\mathrm{Hom}(B^*, B^*)$.

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Flag Condition

Flag Condition

Let $T \in A \otimes B \otimes C$ be a concise tensor. Then if $\underline{R}(T) = m$ there exist $A_1 \subset A_2 \subset \cdots A_m = A^*$ such that $\dim(A_i) = i$ and $\mathbb{P}T_A(A_i) \subset \sigma_i(\mathbb{P}B \times \mathbb{P}C)$.

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Friedland's Normal Form

Theorem ([Friedland], Thm 3.1)

Let $T \in A \otimes B \otimes C$ *be* 1_A*-degenerate and rank of elements of* $T_A(A^*)$ *are bounded by* $m-1$ *but not by* $m-2$*. Then there exist bases of* A, B, C *such that, letting* $X_1, \cdots, X_m$ *be a basis of* $T_A(A^*)$ *as a space of matrices,*

1. $X_1 = \begin{pmatrix} Id_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$
2. $X_m = \begin{pmatrix} \mathbf{x}_m & \omega \\ \alpha & 0 \end{pmatrix}$
3. *For all* $2 \leq s \leq m-1$, $X_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ 0 & 0 \end{pmatrix}$.

Here $\omega \in \mathbb{C}^{m-1}$, $\alpha \in \mathbb{C}^{(m-1)*}$, $\mathbf{x}_m, \mathbf{x}_s \in Mat_{(m-1)\times(m-1)}$.
Moreover, $\alpha \mathbf{x}_m^j \omega = 0$ *for all* j*, and letting* $U_R = \langle \mathbf{x}_m^j \omega | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1}$ *and* $U_L = \langle \alpha \mathbf{x}_m^j | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{(m-1)*}$*. Then* $\mathbf{x}_s U_R = 0 = U_L \mathbf{x}_s$ *for* $2 \leq s \leq m-1$.

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