Tensors of Minimal Border Rank

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Simultaneously Diagonalizable Matrices

Let $M_1, \cdots, M_m$ be m simultaneously diagonalizable $m \times m$ matrices.

Classical Problem

Classify the closure of the space of a finite number of simultaneously diagonalizable matrices.

Known Results

- End Closed Condition: Known since 1960.[Gerstenhaber]
- Flag Condition: Appeared in 1997.[Burgisser, Clausen, Shokrollahi]

Definitions

Let $T \in A \otimes B \otimes C$.
T can be seen as a linear map $T_A : A^* \to B \otimes C$. Similarly T_B and T_C.

Definition

The *rank* of a tensor T, denoted $\mathbf{R}(T)$, is the smallest r such that $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$.

Definition

The *border rank* of a tensor T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that $T = \lim_{\epsilon \to 0} T_\epsilon$ where $\mathbf{R}(T_\epsilon) = r$.

Definition

The tensor T is called *concise* if T_A, T_B and T_C are of full rank.

Questions

Assume $\dim(A) = \dim(B) = \dim(C) = m$ and $T \in A \otimes B \otimes C$.

Question 1

Find equations for the set of border rank at most m tensors.

Question 1 is answered up to dimension m=4.[Friedland]

Assuming T is concise, minimal possible border rank for T is m.

Question 2

Find equations for the set of concise minimal border rank tensors.

Under a natural genericity condition this question is same as characterizing the closure of simultaneously diagonalizable matrices.

Questions

$B \otimes C$ is identified with linear maps from B^* to C, Hom(B^*, C).

Definition

T is called 1_A-*generic*(similarly 1_B and 1_C) if there exist $x \in A^*$ such that $T_A(x)$ is invertible. T is called 1_*-*generic* if it is 1_A, 1_B or 1_C-generic.

If T is 1_*-generic then we reduce to the previous classical problem of classifying the closure of simultaneously diagonalizable matrices.

Question 3

Find equations for the set of concise, 1_*-generic, minimal border rank tensors.

Question 3 is answered for $m = 5$.[Landsberg, Michalek]

Explanation

Let T be concise, 1_A-generic, and $\alpha_1, \cdots, \alpha_m$ be a basis of A^* with $T_A(\alpha_1)$ full rank. Consider $Id_{B^*}, T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m) \in \mathrm{Hom}(B^*, B^*)$.

If T is of rank m then note that $T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m)$ needs to be simultaneously diagonalizable matrices. This is basically consequence of Strassen's equations.

Thus if T is of border rank m then $T_A(\alpha_1)^{-1}T_A(\alpha_2), \cdots, T_A(\alpha_1)^{-1}T_A(\alpha_m)$ has to be in the closure of the space of simultaneously diagonalizable tuple of matrices.

Motivation

- Question 1 is finding equations for the secant variety, $\sigma_m(\mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1})$.
- Complexity Theory: Latest bound on the exponent of matrix multiplication is achieved through Coppersmith-Winograd tensor. Which is a concise minimal border rank tensor.
- Classical Linear Algebra: Closure of simultaneously diagonalizable matrices.
- Algebraic Geometry: Hilbert schemes and Quot Schemes.

Strassen's Equations

Simultaneously diagonalizable $\implies$ Commuting.

Theorem ([Strassen])

Let X_1, X_2 and Y be in $T_A(A^*)$. If T is of minimal border rank then $X_1 \mathit{adj}(Y) X_2 - X_2 \mathit{adj}(Y) X_1 = 0$.

Assuming T 1_*-generic, we can take Y to be of full rank and in particular identity matrix. Then Strassen's Equations precisely reduces to commuting criterion of X_1 and X_2.

These are necessary conditions a tensor must satisfy in order to be of minimal border rank.

Border Apolarity

A new paper, *Apolarity, Border Rank and Multigraded Hilbert Schemes* by Weronika Buczynska and Jaroslaw Buczynski[Buczynska, Buczynski], gives a new series of necessary conditions for minimal border rank.

210-test

$T \in A \otimes B \otimes C$ concise tensor.

$T_C(C^*) \subset A \otimes B$.

Passes 210-test if $\dim((T_C(C^*) \otimes A) \cap (S^2(A) \otimes B)) \geq m$.

Equivalent to previously known $p = 1$ Koszoul flattening equations.

We determined module structure of these equations for small m.

111-test

The tensor T is said to pass 111-test if $\dim((T_A(A^*) \otimes A) \cap (T_B(B^*) \otimes B) \cap (T_C(C^*) \otimes C)) \geq m$.

So if $\dim((T_A(A^*) \otimes A) \cap (T_B(B^*) \otimes B)) < m$ the tensor fails 111-test. These are called two factor 111-tests.

Example

Consider the tensor(thanks to Joachim Jelisiejew) $T_J =$

$$a_1 \otimes (b_1 \otimes c_3 + b_5 \otimes c_5) + a_2 \otimes (b_1 \otimes c_4 + b_5 \otimes c_5) + a_3 \otimes (b_2 \otimes c_3 + b_5 \otimes c_5)$$

$$+a_4 \otimes (b_2 \otimes c_4 + b_5 \otimes c_5) + a_5 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3 + b_4 \otimes c_4 + b_5 \otimes c_5).$$

- T_J satisfies Strassen's equations.
- Known to be not minimal border rank using techniques from Deformation Theory, not polynomial.
- Fails 111-test, polynomial criterion.
- T_J in fact fails two factor 111-test.

Geometric Interpretation

Let $\mathfrak{g}_T := \{x \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C) | x.T = 0\}$ be the symmetry lie algebra of T and $\mathfrak{g}_{AB} := \{x \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) | x.T = 0\}$. Similarly $\mathfrak{g}_{BC}$ and $\mathfrak{g}_{AC}$.

Proposition

If T is of minimal border rank then $\dim(\mathfrak{g}_T) \geq 2m$.

Proposition

The tensor T passes all two factor 111-tests if and only if $\dim(\mathfrak{g}_T) \geq 2m$ and $\dim(\mathfrak{g}_{AB}), \dim(\mathfrak{g}_{BC}), \dim(\mathfrak{g}_{AC}) \geq m$.

Work in Progress

- Understanding geometric significance of 111-test and possibly higher tests.
- Finding the $\mathsf{GL}(A) \times \mathsf{GL}(B) \times \mathsf{GL}(C)$-module structure of the equations originating from these tests.

This is joint work(in progress) with Prof. JM Landsberg.

Towards m=5, 1_A-generic case

In this case T is 1_A-generic and concise.

- Already solved, Question 3 for m=5.[Landsberg, Michalek]
- Known answer: Strassen's equations together with end closed condition.

Theorem (J. Jelisiejew, K. Sivic)

The closure of the space of 4-tuple of 5×5 commuting matrices is not irreducible and has exactly two components.
The principal component is the closure of simultaneously diagonalizable matrices and one other bad component. Further $T_{J_A}(A^*)$ belongs to the bad component.

Work in Progress

- Any general point of the *bad* component fails 111-test and so non minimal border rank. Its boundary falls in the principal component.

- Extending same technique for $m = 6$ and answering Question 3 for $m = 6$. There are 3 *bad* components for $m = 6$.

This is joint work(in progress) with Prof. JM Landsberg and Prof. Joachim Jelisiejew.

m=5, 1-degenerate

Theorem ([Friedland], Thm 3.1)

Let $T \in A \otimes B \otimes C$ *be* 1_C*-degenerate and rank of elements of* $T_C(C^*)$ *are bounded by* $m-1$ *but not by* $m-2$*. Then there exist bases of* A, B, C *such that, letting* $X_1, \cdots, X_m$ *be a basis of* $T_C(C^*)$ *as a space of matrices,*

1. $X_1 = \begin{pmatrix} \mathrm{Id}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$
2. $X_m = \begin{pmatrix} \mathbf{x}_m & e_1 \\ e^{m-1} & 0 \end{pmatrix}$
3. *For all* $2 \leq s \leq m-1$, $X_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ 0 & 0 \end{pmatrix}$.

Here $e_1 = (1, 0, \cdots, 0)^t \in \mathbb{C}^{m-1}$, $e^{m-1} = (0, 0, \cdots, 1) \in \mathbb{C}^{m-1*}$, $\mathbf{x}_j \in Mat_{(m-1)\times(m-1)}$.
Moreover, let $U_R = \langle \mathbf{x}_m^j e_1 | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1}$ *and* $U_L = \langle e^{m-1}\mathbf{x}_m^j | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1*}$*. Then* $e^{m-1}U_R = 0, U_L e_1 = 0$ *and* $\mathbf{x}_s U_R = 0 = U_L \mathbf{x}_s$ *for* $2 \leq s \leq m-1$.

m=5, 1-degenerate

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ x_1^2 & x_2^2 & x_3^2 & 0 & 0 \\ x_1^3 & x_2^3 & x_3^3 & 0 & 0 \\ 0 & x_2^4 & x_3^4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X_s = \begin{pmatrix} 0 & p_s & u_s & x_s & 0 \\ 0 & q_s & v_s & y_s & 0 \\ 0 & r_s & w_s & z_s & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $s = 2, 3, 4$.

Cases to consider

Case: $x_2^3 \neq 0$ and $x_3^4 = 0$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & x_2^2 & x_3^2 & 0 & 0 \\ 0 & x_2^3 & x_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, X_s = \begin{pmatrix} 0 & 0 & 0 & x_s & 0 \\ 0 & 0 & 0 & y_s & 0 \\ 0 & 0 & 0 & z_s & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $s = 2, 3, 4.$

It turns out in this case the tensor is always 1_A-generic.

Cases to consider

Case: $x_2^3 = x_3^4 = 0$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & x_2^2 & x_3^2 & 0 & 0 \\ 0 & 0 & x_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X_s = \begin{pmatrix} 0 & 0 & u_s & x_s & 0 \\ 0 & 0 & v_s & y_s & 0 \\ 0 & 0 & w_s & z_s & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $s = 2, 3, 4$.

This case is not automatic. Needs further analysis.

Cases to consider

Case: $x_1^2 = x_3^4 = 0$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & x_2^2 & x_3^2 & 0 & 0 \\ 0 & x_2^3 & x_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X_s = \begin{pmatrix} 0 & p_s & u_s & x_s & 0 \\ 0 & q_s & v_s & y_s & 0 \\ 0 & r_s & w_s & z_s & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $s = 2, 3, 4$.

This case is not automatic. Needs further analysis.

Work in Progress

- Do the remaining calculations for those cases.
- Extending these techniques using this normal form for dimension 6 and answer the same question for $m = 6$.

This is joint work(in progress) with Prof. JM Landsberg.

Thank you

Thank you!

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