Concise Tensors of Minimal Border Rank

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Oct 7, 2022

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The tensor T is called *concise* if T_A, T_B and T_C are injective.

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Assume $\dim(A) = \dim(B) = \dim(C) = m$ and $T \in A \otimes B \otimes C$.

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Question 2

Find equations for the set of concise minimal border rank tensors.

Under a natural genericity condition this question is same as characterizing the closure of simultaneously diagonalizable matrices.

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Find equations for the set of concise, 1_*-generic, minimal border rank tensors.

Question 3 is answered for $m = 5$.[Landsberg, Michalek]

Simultaneously Diagonalizable Matrices

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Classical Problem

Characterize the closure of the space of a m-tuple of simultaneously diagonalizable matrices.

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Characterize the closure of the space of a m-tuple of simultaneously diagonalizable matrices.

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Let T be 1_A-generic and concise, then it's of minimal border rank $\iff$ $T_A(A^*)T_A(\alpha)^{-1} \subset \text{End}(C)$ is in the closure of the space of simultaneously diagonalizable matrices.

Motivation

- Question 1 is finding equations for the secant variety, $\sigma_m(Seg(\mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1}))$.

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- Classical Linear Algebra: Closure of simultaneously diagonalizable matrices.

- Algebraic Geometry: Hilbert schemes and Quot Schemes.

Salmon Prize Problem

In 2007, E. Allman offered a prize of smoked Alaskan copper river salmon to anyone who could find the defining ideal of the following secant variety:

$$\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$$

 

Strassen's Equations

Simultaneously diagonalizable $\implies$ Commuting.

Theorem ([Strassen])

Let X_1, X_2 and Y be in $T_A(A^*)$. If T is of minimal border rank then $\mathit{adj}(Y)X_1\mathit{adj}(Y)X_2 - \mathit{adj}(Y)X_2\mathit{adj}(Y)X_1 = 0$.

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These are necessary conditions a tensor must satisfy in order to be of minimal border rank.

Towards m=5 and 6, 1_A-generic case

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The closure of the space of 4-tuple of 5×5 commuting matrices is not irreducible and has exactly two components.
The principal component is the closure of simultaneously diagonalizable matrices and one other component.

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Upshot: This extends to $m = 6$ and the same remains true!

New Development

Theorem (Jelisiejew, Landsberg, P)

Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, where $m = 5, 6$, be a concise 1_*-generic tensor. Then T is of minimal border rank if and only if T satisfies Strassen's equations and End Closed condition.

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Outline?

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- Does not extend to $m = 7$. Explicit example in *Abelian Tensors*[Landsberg, Michalek].
- Does not make sense for 1-degenerate tensors.

Another Direction

A new paper, *Apolarity, border rank and multigraded Hilbert scheme* by Weronika Buczynska and Jaroslaw Buczynski[Buczynska, Buczynski], gives new necessary conditions for minimal border rank.

111-test

The tensor T is said to pass 111-test if
$\dim((T_A(A^*) \otimes A) \cap (T_B(B^*) \otimes B) \cap (T_C(C^*) \otimes C)) \geq m.$

Definition

If T satisfies the above inequality then T is called 111-abundant and if it satisfies without excess, i.e. the above inequality becomes equality then we say T is 111-sharp.

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Towards $m = 5$, 1-degenerate case

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Theorem (Jelisiejew, Landsberg, P)

When $m \leq 5$, the set of concise minimal border rank tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is the zero set of the 111-equations.

Things getting wild...

Up to the action of $\mathrm{GL}_5(\mathbb{C})^{\times 3} \rtimes \mathcal{S}_3$ there are exactly 5 concise 1-degenerate minimal border rank tensors in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ and those are:

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$$T_{\mathcal{O}_{58}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ x_5 & x_1 & x_4 & -x_2 & \\ & & x_1 & & \\ & & -x_5 & x_1 & \\ & & & x_5 & \end{pmatrix}, T_{\mathcal{O}_{57}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 & x_4 & -x_2 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix},$$

$$\mathsf{T}_{\mathcal{O}_{56}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 + x_5 & & x_4 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix}, T_{\mathcal{O}_{55}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 & x_5 & x_4 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix},$$

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Definition

The smoothable rank of a tensor $T \in A \otimes B \otimes C$, denoted $\mathbf{S}(T)$, is the minimal degree of a zero dimensional scheme $\mathrm{Spec}(R) \subseteq \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$ such that $T \in \langle \mathrm{Spec}(R) \rangle$.

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In general $\mathbf{R}(T) \geq \mathbf{S}(T) \geq \underline{\mathbf{R}}(T)$. If $\mathbf{S}(T) > \underline{\mathbf{R}}(T)$ then T is called a *wild* tensor.

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Theorem

In $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ concise, minimal border rank, wild tensors are precisely $T_{\mathcal{O}_{58}}, T_{\mathcal{O}_{57}}, T_{\mathcal{O}_{56}}, T_{\mathcal{O}_{55}}, T_{\mathcal{O}_{54}}$.

Thank you!!!

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Algebraic complexity theory

Algorithm

For $T \in A \otimes B \otimes C$ a concise tensor

Definition

$Ann(T) = \{x \in Sym(A^*) \otimes Sym(B^*) \otimes Sym(C^*) | x.T = 0\}$

1. $I \subset Ann(T)$ i.e. $I_{110} \subset T(C^*)^\perp$, etc. and $I_{111} \subset T^\perp$
2. For all i, j, k such that $i + j + k > 1$, then codim $I_{ijk} = m$
3. The image of the multiplication map $I_{i-1,j,k} \otimes A^* \oplus I_{i,j-1,k} \otimes B^* \oplus I_{i,j,k-1} \otimes C^* \to S^i A^* \otimes S^j B^* \otimes S^k C^*$ is contained in I_{ijk}

We refer to the codimension criterion at (i, j, k) grade as ijk-test.

back

Reduction to Diagonalization

Let $T \in A \otimes B \otimes C$ concise, 1_A-generic, and $\alpha_1, \cdots, \alpha_m$ a basis of A^* with $T_A(\alpha_1)$ full rank. Consider $\langle Id_{B^*}, M_2, \cdots, M_m \rangle \subset \text{Hom}(B^*, B^*)$, where $M_i = T_A(\alpha_1)^{-1} T_A(\alpha_i)$.

- T is rank $m \iff$ M_i's simultaneously diagonalizable.
- T is border rank $m \iff$ M_i's are in the closure of the space of simultaneously diagonalizable matrices.

back

Apolarity Lemma

Let $S = \mathbb{C}[x_1, \ldots, x_n]$ and $T = \mathbb{C}[y_1, \ldots, y_n]$. Define T action on S as $y_i \cdot x_j = \frac{\partial}{\partial x_i} x_j$.
Let $f \in S_d$ and $f^\perp := \{t \in T | t \cdot f = 0\}$ be called the apolar ideal of f.

Apolarity Lemma

$f \in \langle l_1^d, \ldots, l_r^d \rangle \iff f^\perp$ contains an ideal of r distinct points.

back

End Closed Condition

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Let $T \in A \otimes B \otimes C$ be a concise 1_A-generic tensor and $\alpha \in A^*$ such that $T_A(\alpha)$ has full rank. Then $T(\alpha)^{-1}T_A(A^*)$ is a subalgebra of $\mathrm{Hom}(B^*, B^*)$.

back

Flag Condition

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Let $T \in A \otimes B \otimes C$ be a concise tensor. Then if $\underline{R}(T) = m$ there exist $A_1 \subset A_2 \subset \cdots A_m = A^*$ such that $\dim(A_i) = i$ and $T_A(A_i) \subset \sigma_i(\mathbb{P}B \times \mathbb{P}C)$.

back

m=5, 1-degenerate

Theorem ([Friedland], Thm 3.1)

Let $T \in A \otimes B \otimes C$ *be* 1_C*-degenerate and rank of elements of* $T_C(C^*)$ *are bounded by* $m-1$ *but not by* $m-2$*. Then there exist bases of* A, B, C *such that, letting* $X_1, \cdots, X_m$ *be a basis of* $T_C(C^*)$ *as a space of matrices,*

1. $X_1 = \begin{pmatrix} Id_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$
2. $X_m = \begin{pmatrix} \mathbf{x}_m & e_1 \\ e^{m-1} & 0 \end{pmatrix}$
3. *For all* $2 \leq s \leq m-1$, $X_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ 0 & 0 \end{pmatrix}$.

Here $e_1 = (1, 0, \cdots, 0)^t \in \mathbb{C}^{m-1}$, $e^{m-1} = (0, 0, \cdots, 1) \in \mathbb{C}^{m-1*}$, $\mathbf{x}_j \in Mat_{(m-1)\times(m-1)}$.
Moreover, let $U_R = \langle \mathbf{x}_m^j e_1 | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1}$ *and* $U_L = \langle e^{m-1} \mathbf{x}_m^j | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1*}$. *Then* $e^{m-1} U_R = 0, U_L e_1 = 0$ *and* $\mathbf{x}_s U_R = 0 = U_L \mathbf{x}_s$ *for* $2 \leq s \leq m-1$.

back

Rank vs Border Rank

Consider: $T = (e_1 \otimes e_1 \otimes e_2) + (e_1 \otimes e_2 \otimes e_1) + e_2 \otimes e_1 \otimes e_1) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Rank 3 as $\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$, $a, b \in \mathbb{C}$ can not be written as sum of two rank one matrices.

Border rank 2 as $T(\epsilon) = \frac{1}{\epsilon}[(e_1 + \epsilon e_2) \otimes (e_1 + \epsilon e_2) \otimes (e_1 + \epsilon e_2) - e_1 \otimes e_1 \otimes e_1]$ and $T(\epsilon) \to T$ as $\epsilon \to 0$.

back

Conciseness Restated

Let $T \in A \otimes B \otimes C$ and $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$ be a rank decomposition.

Definition

T is *concise if* $\{a_1, \cdots, a_r\}$ *spans* A*,* $\{b_1, \cdots, b_r\}$ *spans* B*, and* $\{c_1, \cdots, c_r\}$ *spans* C*.*

back

Secant Variety

Working over $\mathbb{C}$.
The join of two varieties $Y, Z \subset \mathbb{P}V$ is $J(Y,Z) = \overline{\cup_{x\in Y, y\in Z, x\neq y}\mathbb{P}^1_{xy}}$.
The r-th secant variety of a variety $X \subset \mathbb{P}V$ is

$$\sigma_r(X) = \overline{\cup_{P_1,\cdots,P_r\in X}\langle P_1,\cdots,P_r\rangle} = J(Y, J(Y,\cdots))$$

Fact: Joins and Secants of irreducible varieties are irreducible.
Fact: $X = Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$, then Euclidean and Zariski closure of X agree for $\sigma_r(X)$.
Sketch:

- Euclidean closure is contained in Zariski closure.
- Z an irreducible variety and $U \subset Z$ a Zariski open subset then $\overline{U} = Z$ in terms of Zariski closure and Euclidean closure.
- For X take U to be set of rank at most r tensors.

Secant Variety

Expected Dimension:

- For $\sigma_r(X)$ is $\min\{rn + r - 1, N\}$, where $X \subset \mathbb{P}^N$ and $\dim(X) = n$.

Defect $\delta_r := rn + r - 1 - \dim \sigma_r(X)$.

Theorem (Terracini's Lemma)

Let $P_1, \cdots, P_r \in X$ be general points and $P \in \langle P_1, \cdots, P_r \rangle \subset \sigma_r(X)$ a general point. Then

$$T_P(\sigma_r(X)) = \langle T_{P_1}(X), \cdots, T_{P_r}(X) \rangle$$

Consequence: $\delta_r(Seg(\mathbb{P}A \times \mathbb{P}B)) = r^2 - r$

Consequence: $m > 2$ and $r \leq \min\{\dim V_i\}$ then $\sigma_r(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_m))$ is not defective.

back

Outline of the Proof

Fact: For a finite algebra $\mathcal{A} = \prod \mathcal{A}_t$, with $\mathcal{A}_t$ local. Algebra $\mathcal{A}$ can be generated by q elements if and only if $H_{\mathcal{A}_t}(1) \leq t$ for all t.

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For m=5: Let T be 1_A-generic with $T_A(\alpha_0)$ full rank and
$E = T_A(A^*)T_A(\alpha_0)^{-1} \subset \text{End}(C)$ space of commuting matrices.
Gives C an $S := \mathbb{C}[y_1, \cdots, y_4]$-module structure.
$S/Ann(C) \cong E = \langle x_1, \cdots, x_5 \rangle$

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Case-I: *E contains an element with more than one eigenvalue.*
That implies(by Lemma 3.12 of Components and Singularities paper)
$S/Ann(C) \cong \prod_t S/K_t$, non-trivial product of local algebras and
$\dim(E) = 5 \implies \dim(S/K_t) \leq 4 \implies H_{S/K_t}(1) \leq 3$. Thus E can be generated by at most 3 matrices.

Outline of the Proof

Fact: For a finite algebra $\mathcal{A} = \prod \mathcal{A}_t$, with $\mathcal{A}_t$ local. Algebra $\mathcal{A}$ can be generated by q elements if and only if $H_{\mathcal{A}_t}(1) \leq t$ for all t.

For m=5: Let T be 1_A-generic with $T_A(\alpha_0)$ full rank and
$E = T_A(A^*)T_A(\alpha_0)^{-1} \subset \mathrm{End}(C)$ space of commuting matrices.
Gives C an $S := \mathbb{C}[y_1, \cdots, y_4]$-module structure.
$S/Ann(C) \cong E = \langle x_1, \cdots, x_5 \rangle$

Case-I: *E contains an element with more than one eigenvalue.*
That implies(by Lemma 3.12 of Components and Singularities paper)
$S/Ann(C) \cong \prod_t S/K_t$, non-trivial product of local algebras and
$\dim(E) = 5 \implies \dim(S/K_t) \leq 4 \implies H_{S/K_t}(1) \leq 3$. Thus E can be generated by at most 3 matrices.

Case-II:*All elements of E are nilpotent.*
Then $H_E(0) = 1,\ H_E(1) \geq 4 \implies H_E(2) = 0$. Then by Thm 6.14 of Components and Singularities of Commuting Matrices-J,S paper this tuple is in the closure of the tuple of simultaneously diagonalizable matrices.

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Friedland's Normal Form

Theorem ([Friedland], Thm 3.1)

Let $T \in A \otimes B \otimes C$ be 1_A-degenerate and rank of elements of $T_A(A^*)$ are bounded by $m-1$ but not by $m-2$. Then there exist bases of A, B, C such that, letting $X_1, \cdots, X_m$ be a basis of $T_A(A^*)$ as a space of matrices,

1. $X_1 = \begin{pmatrix} Id_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$
2. $X_m = \begin{pmatrix} \mathbf{x}_m & \omega \\ \alpha & 0 \end{pmatrix}$
3. *For all* $2 \leq s \leq m-1$, $X_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ 0 & 0 \end{pmatrix}$.

Here $\omega \in \mathbb{C}^{m-1}$, $\alpha \in \mathbb{C}^{(m-1)*}$, $\mathbf{x}_m, \mathbf{x}_s \in Mat_{(m-1)\times(m-1)}$.
Moreover, $\alpha \mathbf{x}_m^j \omega = 0$ *for all* j, *and letting* $U_R = \langle \mathbf{x}_m^j \omega | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{m-1}$ *and* $U_L = \langle \alpha \mathbf{x}_m^j | j \in \mathbb{Z}_{\geq 0} \rangle \subset \mathbb{C}^{(m-1)*}$. *Then* $\mathbf{x}_s U_R = 0 = U_L \mathbf{x}_s$ *for* $2 \leq s \leq m-1$.